West Bengal State University

B.A./B.Sc./B.Com. (Honours, Major, General) Examinations, 2012

PART–III

MATHEMATICS — Honours

Paper–V

Duration : 4 Hours Full Marks : 1

Candidates are required to give their answers in their own words as far as practicable.

The figures in the margin indicate full marks.

GROUP - A

(Marks : 70)

Answer Question No. 1 and any *five* from the rest.

1. Answer any *five* of the following questions : $5 \times 3 =$

a) Let $S = \{ r \in Q : \sqrt{2} < r < \sqrt{3} \}$. Show that S is not compact in Q, the set rationals.

b) Find the radius of convergence of the power series :

$$x + \frac{(2!)^2}{4!} x^2 + \frac{(3!)^2}{6!} x^3 + \ldots..$$

c) Prove that the series $\sum \frac{2^n x^{2^n - 1}}{1 + x^{2^n}}$, $-\frac{1}{2} \le x \le \frac{1}{2}$ converges uniformly on $\left[-\frac{1}{2}, \frac{1}{2} \right.$

d) Prove that if the sequence $\{f_n\}_n$ converges pointwise to a function f on a fin set then the convergence is uniform.

e) If $\log_e x = \int_1^x \frac{dt}{t}$, $x > 0$, then show that $\log_e x$ is strictly increasing on (0, and $\log_e x \to \infty$ and $x \to \infty$.

f) Show that the intrinsic equation of $ay^2 = x^3$, where the arc length is measured from the origin is $27s = 8a\left\{\sec^3\psi - 1\right\}$.

g) Prove or disprove : If a function $f : [\,a,\,b\,] \rightarrow \mathbb{R}$ is of bounded variation on $[\,a,\,b\,]$ then f is continuous on $[\,a,\,b\,]$.

h) Give example with justification of Riemann integrable function which has no primitive.

i) Let $S \subset \mathbb{R}^2$ and $f : S \rightarrow \mathbb{R}$ be a function of two independent variables x and y such that f has continuous partial derivatives of first order in some nbd.$N(\,a,\,b\,)$ of a point $(\,a,\,b\,) \in S$. Then show that

$f(\,a + h,\,b + k\,) = f(\,a,\,b\,) + hf_x(\,a + \theta h,\,b + \theta k\,) + kf_y(\,a + \theta h,\,b + \theta k\,)$, where $0 < \theta < 1$, and $(\,a + h,\,b + k\,) \in N(\,a,\,b\,) \cap S$.

2. a) If S is a closed and bounded set of real numbers, then prove that every open cover of S has a finite subcover. 5

b) Let A be a compact set in $\mathbb{R}$ such that every point of A in an isolated point of A. Show that A is finite. 3

c) If A and B are two open sets in $\mathbb{R}$ such that $A \cap B$ is compact, then show that $A \cap B = \Phi$. 3

3. a) State and prove the Cauchy criterion for uniform convergence of infinite series of functions defined on a set of real numbers. 4

b) Prove that $\left\{f_n(x)\right\}_n = \left\{\dfrac{x}{1+nx^2} : x \text{ real}\right\}$ converges uniformly on any closed interval $I \equiv [\,a,\,b\,]$. 3

c) If $\sum\limits_{n=1}^{\infty} f_n(x)$ be uniformly convergent on $[\,a,\,b\,]$ and $g : [\,a,\,b\,] \rightarrow R$ be a bounded function on $[\,a,\,b\,]$, show that $\sum\limits_{n=1}^{\infty} f_n(x)\,g(x)$ is uniformly convergent on $[\,a,\,b\,]$. 4

4. a) Prove that a power series $\sum_{n=0}^{\infty} a_n x^n$, with radius of convergence ρ, conve

absolutely and uniformly on $[a, b]$ if $[a, b] \subset (-\rho, \rho)$.

b) If power series $\sum_{n=0}^{\infty} a_n x^n$, with radius of convergence ρ, converges to

function f for $|x| < \rho$, then show that $a_n = \frac{f^n(0)}{n!}$, $n = 1, 2, 3, \ldots$.

c) Starting from the power series expansion for $\frac{1}{1+x^2}(|x| < 1)$ derive the p

series of $\tan^{-1} x$. From this obtain the sum of the infinite s

$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots$

5. a) If a function $f : [a, b] \to \mathbb{R}$ has a bounded derivative on $[a, b]$, then show

f is of bounded variation on $[a, b]$. Give an example (justification

necessary) to show that the converse may not be true.

b) Let $f, g : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{1}{3^n} & \text{if } \frac{1}{3^{n+1}} < x < \frac{1}{3^n};\ n = 0, 1, 2, \ldots\ldots \\ 0 & \text{if } x = 0 \end{cases}, \quad g(x) = \begin{cases} x^3 \sin \frac{1}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Prove that the curve $\gamma = (f, g)$ is rectifiable.

c) Find the length of one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)(a$

6. a) Let $f : [a, b] \to \mathbb{R}$ be a function such that $|f(x)| < k\ (k > 0)\ \forall x \in [a, b]$

be a partition of $[a, b]$ having norm $< \delta$. If Q is a refinement of P having e

one more point of division, then prove that $0 \le U(p, f) - U(Q, f) < 2k\delta$.

b) Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Prove that for every $\in > 0$,

corresponds $\delta > 0$, such that $U(\rho, f) < \int_a^{-b} f + \in$, for all partitions P of [

with norm $< \delta$.

10. a) Evaluate $\iint_E \left(x^2 + y^2\right) dx\, dy$ over the region E bounded by $xy = 1$, $y = 0$, y … $x = 2$.

b) Show, by changing the order of integration, that $\int_0^1 dx \int_x^{1/x} \frac{y\, dy}{(1+xy)^2\left(1+y^2\right)} = \frac{\pi -}{4}$…

c) Prove, with proper justification, that $\int_0^a \frac{\log(1+ax)}{1+x^2}\, dx = \frac{1}{2}\log\left(1+a^2\right)\tan^{-1}$… ($a > 0$), and hence show that $\int_0^1 \frac{\log(1+x)}{1+x^2}\, dx = \frac{\pi}{8} \log 2$.

GROUP - B

(Marks : 15)

Answer any *one* of the following :

11. a) Let the function $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined as $d(x, y) = |x_1 - y_1| + |x_2 - y_2|$ … $x, y \in \mathbb{R}^2$, where $x = (x_1, x_2)$, $y = (y_1, y_2)$. Show that d is a metric on $\mathbb{R}^2$.

b) Let (x, d) be a metric space. Prove that

i) union of an arbitrary family of open sets in X is open in X

ii) a non-empty subset G of X is open iff it is a union of open balls.

c) i) For a point x and a subset A of a metric space (X, d), prove that x is … limit point of A iff for every neighbourhood U of x, $U \cap A$ is an infinite se…

ii) Let (X, d) be a metric space and $A \subseteq X$. Prove that A° is an open s… contained in A and $\overline{A}$ is a closed set containing A. (symbols have the usual meanings)

3 +

12. a) Prove that every convergent sequence in a metric space is a Cauchy sequence. Also prove that every Cauchy sequence is bounded but not conversely. 4

b) Show that the function space $C[a, b]$, consisting of all real valued continuous functions on $[a, b]$ is a complete metric space with the metric d, given by

$$d(f, g) = \sup\{\,|f(x) - g(x)| : a \le x \le b\,\}, \text{ for all } f, g \in C[a, b].$$ 6

c) Let Y be a subspace of a metric space (X, d), prove that

i) $G\,(\subseteq Y)$ is open in the metric space (Y, d_Y) iff $G = H \cap Y$, for some open set H in (X, d).

ii) $F\,(\subseteq Y)$ is closed in (Y, d_Y) iff $F = V \cap Y$, for some closed set V in (X, d). 5

GROUP - C

(Marks : 15)

Answer any *one* of the following.

13. a) Define extended complex plane $\mathbb{C}_\infty$. How do you represent $\mathbb{C}_\infty$ geometrically on a sphere. 1 + 3

b) Prove that a function $f : D \to \mathbb{C}$ is continuous over D iff $f^{-1}(G)$ is open in D for every open set G in $\mathbb{C}$. 6

c) Examine differentiability of $f : \mathbb{C} \to \mathbb{C}$ where,

i) $f(z) = z^2,\ z \in \mathbb{C}$

ii) $f(z) = |z|^2,\ z \in \mathbb{C}$. 5

14. a) Let G be a region in $\mathbb{C}$ and $f : G \to \mathbb{C}$ be analytic such that $|f|$ is constant G. Prove that f is constant on G.

b) If u and v are the real and imaginary parts of an analytic function f and $u(x, y) = x^3 - 3xy^2$, find $f(z)$.

c) Show that the following function f is not differentiable at the origin even thou it satisfies Cauchy-Riemann equations there :

$$f(x + iy)\begin{cases} = \dfrac{xy^2(x+iy)}{x^2+y^4}, & (x+iy) \neq 0 \\ = 0, & (x+iy) = 0 \end{cases}$$