MTMA(HN)-01 156

West Bengal State University

B.A./B.Sc./B.Com (Honours, Major, General) Examinations, 2014

PART - I

MATHEMATICS — HONOURS

Paper - I

Duration : 4 Hours] [Full Marks : 100

The figures in the margin indicate full marks.

GROUP - A

Answer any *five* questions. $5 \times 5 = 25$

1. i) State first principle of Mathematical Induction. 2

   ii) Using the first principle of Mathematical Induction prove that $2^{2n+1} - 9n^2 + 3n - 2$ is divisible by 54. 3

2. i) If a and b are two positive integers such that g.c.d. of a and b is d, then prove that there exist integers s and t such that $d = s.a + t.b$. 3

   ii) If p is a prime number and a, b are positive integers such that $p | ab$, then prove that either $p | a$ or $p | b$. 2

3. i) State Euler's function $\phi(n)$, where n is a positive integer. 1

   ii) If m and n are two positive integers such that m is prime to n, then show that $\phi(mn) = \phi(m)\phi(n)$. 4

4. If $i^{i^{\cdots \text{to} \infty}} = A + iB$, prove that $\tan\frac{\pi A}{2} = \frac{B}{A}$ and $A^2 + B^2 = e^{-\pi B}$, by considering the principal value only. 4 + 1

5. If $i^{p+iq} = p + iq$, show that $p^2 + q^2 = e^{-(4n+1)\pi q}$, considering general values only, where $n = 0, \pm 1, \pm 2, \ldots$. 5

6. Show that $\sin^4\theta\cos^5\theta = \frac{1}{256}(\cos 9\theta + \cos 7\theta - 4\cos 5\theta - 4\cos 3\theta + 6\cos\theta)$. 5

7. i) If $x^4 + px^2 + qx + r$ has a factor of the form $(x-\alpha)^3$, then show that $8p^3 = -27q^2$ and $p^2 = -12r$. 2 + 2

ii) Express the polynomial $x^4 + 3x^3 + 5x^2 + 3x + 1$ as a polynomial in $x + 2$. 1

8. i) If the equation $x^n - px^2 + r = 0$ has two equal roots, show that $n^n r^{n-2} = 4p^n (n-2)^{n-2}$. 2

ii) If α, β, γ be the roots of the equation $x^3 + px^2 + qx + r = 0$, then prove that $\sum \alpha^8 = 2q^4 - 8qr^2$. 3

9. Use Sturm's function to find out the location of the roots of the equation $x^3 + x^2 - 2x - 1 = 0$. 5

GROUP - B

Answer any *two* questions. $2 \times 10 = 20$

10. i) Let A, B, C be three non-empty subsets of a set S. Then prove that $(A-B)\times C=(A\times C)-(B\times C)$. 3

ii) Let R be the set of all real numbers and $(-1, 1)$ be the interval defined by $(-1,1)=\{x\in R:-1<x<1\}$. Prove that the mapping $f: R \to (-1, 1)$ defined by $f(x)=\dfrac{x}{1+|x|}, \forall x\in R$ is one to one and onto. 2 + 3

iii) Let $S=\{a,b,c,d\}$. Find a relation ρ on S which is reflexive, symmetric but not transitive. 2

11. i) Let N be the set of all positive integers. Let R be the relation on N defined by $R=\{(a,b)\in Z\times Z: a-b\le 0\}$. Prove that R is a partial order relation on N. 3

ii) Let A be a non-empty set and $f: A \to A$ be a one-one mapping. Then prove that $f^n: A\to A$ is a one-one mapping for all integers $n\ge 1$, where $f^n: A\to A$ is defined as follows :

$$f'(x)=f(x),\ f^{1+n}(x)=(f\,o\,f^n)(x),$$

for all $x\in A$, and n is any positive integer. 3

Let $(G,*)$ be a group and $a,b \in G$ such that $a^{-1} * b^2 * a = b^3$ and $b^{-1} * a^2 * b = a^3$. Show that $a = b = e$. 4

Let $(S,*)$ be a finite semigroup in which both Cancellation laws hold. Prove that $(S,*)$ is a group. 3

Let R be the set of all real numbers. Let $G = \{(a,b) : a,b \in R, b \neq 0\}$. Define a binary operation $*$ on G by $(a,b)*(c,d) = (a+bc, bd)$ for all $(a,b),(c,d) \in G$. Show that $(G,*)$ is a non-commutative group. 3

Let $(H,*)$ and $(K,*)$ be two subgroups of a group $(G,*)$. Then prove that $H*K$ is a subgroup of $(G,*)$ if and only if $H*K = K*H$, where $H*K = \{h*k : h \in H, k \in K\}$. 4

Prove that a ring R is commutative if and only if $(a+b)^2 = a^2 + 2ab + b^2$ for all $a,b \in R$. 2

If R is an integral domain of prime characteristic p, then prove that $(a+b)^p = a^p + b^p$. 4

Prove that the ring of matrices $\left\{ \begin{pmatrix} a & b \\ 2b & a \end{pmatrix} : a,b \in Q \right\}$ is a field, where Q is the set of all rational numbers. 4

GROUP - C

Answer any *three* questions. $3 \times 5 = 15$

14. If A is a unitary matrix and $I + A$ is non-singular, then prove that $(I+A)^{-1}(I-A)$ is Skew-Hermitian, where I is the identity matrix of suitable order. 5

15. Use Laplace's method to show that

$$\begin{vmatrix} 1+x^2 & x & 0 & 0 \\ x & 1+x^2 & x & 0 \\ 0 & x & 1+x^2 & x \\ 0 & 0 & x & 1+x^2 \end{vmatrix} = 1+x^2+x^4+x^6+x^8$$ 5

16. Solve by Cramer's rule :

$$x+y+z=1$$

$$ax+by+cz=k$$

$$a^2x+b^2y+c^2z=k^2,\ a \neq b \neq c.$$ 5

17. Express the matrix $A=\begin{pmatrix} 2 & 0 & 1 \\ 3 & 3 & 0 \\ 6 & 2 & 3 \end{pmatrix}$ as product of elementary matrices and hence, find A^{-1}. 3 + 2

18. i) Test whether $5x^2+y^2+5z^2+4xy-8xz-4yz$ is positive definite. 2

ii) Reduce $\begin{pmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{pmatrix}$ to row-reduced Echelon form and find its rank. 3

19. i) If $\Delta = \begin{vmatrix} h & a & 0 \\ \frac{1}{h} & \frac{1}{b} & \frac{1}{f} \\ 0 & c & f \end{vmatrix}$ and $\Delta' = \begin{vmatrix} \frac{1}{bc}-\frac{1}{f^2} & -\frac{1}{ch} & \frac{1}{fh} \\ af & -fh & ch \\ \frac{1}{fh} & -\frac{1}{af} & \frac{1}{ab}-\frac{1}{h^2} \end{vmatrix}$, then show that

$\frac{\Delta'}{\Delta^2} = -\frac{1}{cafh}$. 3

ii) Show that any square matrix is expressible uniquely as a sum of a symmetric and a skew-symmetric matrix. 2

GROUP - D

Answer any *one* question. 1 × 10 = 10

20. i) A person requires 10, 12 and 12 units of chemicals *A, B* and *C* respectively. A liquid product contains 3, 2 and 1 unit of *A, B* and *C* respectively. A dry product contains 1, 2 and 4 units of *A, B* and *C* per packet. If the liquid product sells for Rs. 2 per jar and the dry product sells for Re. 1 per packet, then formulate the problem as a linear programming problem. 5

ii) Use graphical method to solve the following LPP : 5

Maximize $Z = 2x_1 + 3x_2$

subject to $x_1 + x_2 \leq 30$

$x_2 \geq 3$

$x_2 \leq 12$

$x_1 \leq 20$

$x_1 - x_2 \geq 0$

and $x_1, x_2 \geq 0$.

21. i) Find all the basic feasible solutions of the system of equations : 5

$2x_1 + 6x_2 + 2x_3 + x_4 = 3$

$6x_1 + 4x_2 + 4x_3 + 6x_4 = 2$.

ii) Show that basic feasible solutions of an L.P.P. are linearly independent. 5

GROUP - E

Section - I

Answer any *three* questions. $3 \times 5 = 15$

22. Show that the equation $7x^2 - 2xy + 7y^2 + 22x - 10y + 7 = 0$ represents an ellipse. Find its centre, the equation of axes and directrices of the ellipse. 5

23. Show that, if one of the bisectors of the angles between the pair of straight lines $ax^2 + 2hxy + by^2 = 0$ passes through the point of intersection of two straight lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, then $h(g^2 - f^2) = fg(a - b)$. 5

24. A triangle has the lines $ax^2 + 2hxy + by^2 = 0$ for two of its sides and the point (p, q) for its orthocentre. Show that the equation of its third side is $(a+b)(px + qy) = aq^2 + bp^2 - 2hpq$. 5

25. Find the polar equation of the chord joining two points on the conic $\frac{l}{r} = 1 - e\cos(\theta - \gamma)$ with $\alpha - \beta$ and $\alpha + \beta$ as their vectorial angles. Hence find the equation of the tangent to the conic at $\theta = \alpha$. 5

26. i) Find the angle through which axes must be turned so that $ax^2 + 2hxy + by^2$ becomes an expression of the form $Ax^2 + By^2$. 2

    ii) Show that the condition that the straight line $\frac{1}{r} = a\cos\theta + b\sin\theta$ may touch the circle $r = 2k\cos\theta$ is $b^2k^2 + 2ak = 1$. 3

Section - II

Answer any *three* questions. $3 \times 5 = 15$

27. i) Show that the equation of the plane through the intersection of the planes $x-2y+3z+4=0$ and $2x-3y+4z=7$ and the point (1, –1, 1) is $9x-13y+17z=39$. 3

ii) Find the equation of the plane through (1, 2, 3) and parallel to the plane $3x+4y-5z=0$. 2

28. Prove that the straight lines $\frac{x-1}{2}=\frac{y+1}{-3}=\frac{z+10}{8}$ and $\frac{x-4}{1}=\frac{y+3}{-4}=\frac{z+1}{7}$ intersect and find the plane through them. Also find their point of intersection. 5

29. Find the length of the shortest distance between the straight lines $\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}$ and $\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}$. Find also the equation of the line of shortest distance. 5

30. Show that the point $O\left(-\frac{1}{2},2,0\right)$ is the circumcentre of the triangle formed by the points $P(1,1,0)$, $Q(1,2,1)$ and $R(-2,2,-1)$. 5

31. Prove that the straight lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $4x-3y+1=0=5x-3z+2$ are coplanar. Find also the equation of the plane containing them. 5

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Part-I

MATHEMATICS – Honours

Paper– II

Duration : 4 Hours Full Marks : 100

The figures in the margin indicate full marks.

Group - A

(Marks : 25)

Answer any *five* questions of the following. $5 \times 5 = 25$

1. a) Show that the set Q of rational numbers is dense and Archimedean. 1 + 2

   b) Let A and B be two non-empty bounded sets of real numbers; $a = sup\ A$, $b = sup\ B$.

   Let $C = \{ x+y : x \in A, y \in B \}$.

   Show that $sup\ C = a + b$. 2

2. State and prove Cantor's theorem on nested intervals. 5

3. a) Show that a monotonic increasing sequence which is bounded above is convergent. 3

b) Find the derived set of the set, $S=\left\{ \frac{1}{n}, n \in N \right\}$. 2

4. a) Show that every bounded sequence has a convergent subsequence. 3

b) Show that the sequence $\left\{ x_n \right\}_n$, where $x_n = 1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}$ is not convergent. 2

5. a) Use Cauchy's general principle of convergence to show that the sequence $\left\{ x_n \right\}$ where $x_n = 1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}$ is not convergent. 3

b) Show that the sequence $\left\{ \frac{n}{n+1} \right\}_n$ is a Cauchy sequence. 2

6. a) Prove that the union of two denumerable sets is denumerable. 2

b) Show that the set $I\!R$ of real numbers is not denumerable. 3

7. a) Show that the interior of a set is an open set. 2

b) Show that the necessary and sufficient condition for a set to be closed is that its complement is open. 3

8. a) Show that $\lim_{x \to \infty} \frac{[x]}{x} = 1$, where $[x]$ has its usual meaning. 2

b) Use Sandwich theorem to show that

$$\lim_{x \to 0} x \cos \frac{1}{x} = 0.$$ 3

9. a) Prove that the Dirichlet's function f defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} -1, & \text{when } x \text{ is rational} \\ 1, & \text{when } x \text{ is irrational,} \end{cases}$$

is discontinuous at every point. 2

b) If a function $f : [a, b] \to \mathbb{R}$ is monotone on $[a, b]$ then prove that the set of points of discontinuities of f in $[a, b]$ is a countable set. 3

Group - B

(Marks : 20) .

10. Answer any *two* of the following questions. $2 \times 4 = 8$

a) If $I_{m,n} = \int_0^{\frac{\pi}{2}} \cos^m x \sin nx \, dx$ (m, n are positive integers), show that

$$I_{m,n} = \frac{1}{m+n} + \frac{m}{m+n} I_{m-1, n-1}.$$ 4

b) Show that

$$\int_0^1 \frac{1}{(1-x^n)^{\frac{1}{n}}}\,dx = \frac{\pi}{n}\,\text{cosec}\left(\frac{\pi}{n}\right),\ (n>1).$$ 4

c) For $m > -1,\ n > -1$, prove that

$$\int_a^b (x-a)^m (b-x)^n\,dx = (b-a)^{m+n-1}\,\frac{\Gamma(m+1)\,\Gamma(n+1)}{\Gamma(m+n+2)}.$$ 4

11. Answer any *three* questions : $3 \times 4 = 12$

a) Determine the pedal equation of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with respect to a focus, where $a^2 > b^2$. 4

b) Find the evolute of the curve $x = a(1+\cos^2 t)\sin t,\ y = a\sin^2 t\cos t$. 4

c) Find the asymptotes of the curve $r = a\sec\theta + b\tan\theta$. 4

d) Show that the points of inflection of the curve $y(x^2+a^2) = a^2 x$ lie on a straight line. 4

e) If p_1 and p_2 be the radii of curvature at the ends of two conjugate diameters of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, prove that

$$\left(p_1^{2/3}+p_2^{2/3}\right)(ab)^{2/3} = a^2+b^2.$$ 4

Group - C

(Marks : 30)

Answer any *three* of the following questions. 3 × 10 = 30

12. a) If the equation $Mdx + Ndy = 0$ has one and only one solution, then prove that there exists an infinite number of integrating factors. 2

b) Solve : $(y^2 e^{xy^2} + 4x^3)\,dx + (2xye^{xy^2} - 3y^2)\,dy = 0$. 3

c) Examine whether the equation $(\cos y + y\cos x)\,dx + (\sin x - x\sin y)\,dy = 0$ is exact or not and then solve it. 5

13. a) Show that if y_1 and y_2 be solutions of the equation $\frac{dy}{dx} + P(x)\,y = Q(x)$ and $y_2 = y_1 z$, then $z = 1 + ae^{-\int (Q/y_1)\,dx}$, where a is a constant. 4

b) Reduce the differential equation $y = 2px - p^2 y$ to Clairaut's form by the substitution $x = u$, $y^2 = v$ and then obtain complete primitive and singular solution, if any. 6

14. a) Solve : $\frac{d^2y}{dx^2} - y = xe^x \sin x$. 5

b) By the method of undetermined co-efficients solve the equation

$$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 6y = (x-2)e^x.$$ 5

15. a) Solve the equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \frac{e^x}{1+e^x}$, by using the method of variation of parameters. 6

b) Show that the system of co-axial parabolas $y^2 = 4a(x+a)$ is self orthogonal. 4

16. a) Solve $\left(1-x^2\right)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x(1-x^2)^{3/2}$, given that $y = x$ is a solution of the corresponding homogeneous equation. 5

b) Solve : $(2+x)^2\frac{d^2y}{dx^2} - 4(x+2)\frac{dy}{dx} + 6y = x$. 5

a) Solve the differential equation

$$x\frac{d^2y}{dx^2}+(x-1)\frac{dy}{dx}-y=x^2$$, by the method of operational factors. 5

b) Solve the equation

$$\left(\frac{d^2y}{dx^2}+y\right)\cot x+2\left(\frac{dy}{dx}+y\tan x\right)=\sec x$$ by reducing it to normal form. 5

Group - D

(Marks : 25)

Answer any *five* of the following questions. 5 × 5 = 25

Show, by vector method, that the join of the middle points of two sides of a triangle is parallel to the third side and is half of its length. 5

Given $\vec{a}=\hat{i}+\hat{j}-\hat{k}$, $\vec{b}=2\hat{i}+5\hat{j}-7\hat{k}$. Express $\vec{b}$ in the form $\vec{b}=\vec{c}-\vec{d}$, where $\vec{c}$ is parallel to $\vec{a}$ and $\vec{d}$ is perpendicular to $\vec{a}$. 5

Show that the four points $A(\vec{a}), B(\vec{b}), C(\vec{c}), D(\vec{d})$ are coplanar if and only if $[\vec{b}\ \vec{c}\ \vec{d}]+[\vec{c}\ \vec{d}\ \vec{a}]+[\vec{a}\ \vec{b}\ \vec{d}]=[\vec{a}\ \vec{b}\ \vec{c}]$. 5

21. a) Forces $\vec{P}, \vec{Q}$ act at O and have a resultant $\vec{R}$. If any transversal cuts the lines of action of $\vec{P}, \vec{Q}$ and $\vec{R}$ at A, B, C respectively, then show that

$$\frac{P}{OA}+\frac{Q}{OB}=\frac{R}{OC}, \text{ where } P=|\vec{P}|, Q=|\vec{Q}| \text{ and } R=|\vec{R}|.$$ 3

b) A particle acted on by constant forces $(4\hat{i}+5\hat{j}-3\hat{k})$ and $(3\hat{i}+2\hat{j}+4\hat{k})$ is displaced from the point $(\hat{i}+3\hat{j}+\hat{k})$ to the point $(2\hat{i}-\hat{j}-3\hat{k})$. Find the total work done by the forces. 2

22. Find the position vector of the point of intersection of the straight line joining the points $(\hat{i}+\hat{j}+\hat{k})$ and $(3\hat{i}+2\hat{j}-\hat{k})$ with the plane $\vec{r}\cdot(\hat{k}-\hat{j})=5$. 5

23. Prove that Curl Curl $\vec{f}=\vec{\nabla}(\vec{\nabla}\cdot\vec{f})-\nabla^2\vec{f}$. 5

24. Using vector method, find the shortest distance between the lines through $(6, 2, 2)$ and $(-4, 0, 1)$ and parallel to the vectors $(1, -2, 2)$ and $(3, -2, -2)$ respectively. Find also the points at which the lines meet the common perpendicular. 5

a) Prove that the necessary and sufficient condition for a vector $\vec{r} = \vec{f}(t)$ to have constant magnitude is $\vec{f} . \frac{d\vec{f}}{dt} = 0$. 3

b) Evaluate $\frac{d}{dt}\left(\vec{V} . \frac{d\vec{V}}{dt} \times \frac{d^2\vec{V}}{dt^2} \right)$. 2

a) Show that the vector $\vec{F} = (2x - yz)\hat{i} + (2y - zx)\hat{j} + (2z - xy)\hat{k}$ is irrotational. 2

b) Find the directional derivative of the function $f(x, y, z) = 2xy - z^2$ at the point $P(2, -1, 1)$ in the direction towards the point $(3, 1, -1)$. In what direction is the directional derivative maximum ? 3